

# Derivative without Limit

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The derivative of a real, continuous function (and, in fact, of a more general one) can be defined in a way which is intuitively and geometrically suggestive but avoids limit of a difference quotient. © 1986 Academic Press, Inc.

## 1

**DEFINITION.** A *direction* of a real function  $f$  at a real point  $\xi$  is a real number  $d$  such that in every open interval containing  $\xi$  there are  $a, b$  with

$$a < \xi < b, \quad [f(b) - f(a)]/(b - a) = d,$$

namely, the chord joining the points  $(a, f(a))$ ,  $(b, f(b))$  has slope  $d$ .

Let  $f$  be a real function, differentiable at a real  $\xi$  (by which we mean that  $f'(\xi)$  exists and is finite). Then, as is well known,

$$\lim_{\substack{x \rightarrow \xi \\ y \rightarrow \xi}} [f(y) - f(x)]/(y - x) = f'(\xi). \quad (1)$$

For, given  $\varepsilon > 0$ , let  $\delta > 0$  be such that

$$|[f(x) - f(\xi)](x - \xi)^{-1} - f'(\xi)| < \varepsilon$$

whenever  $0 < |x - \xi| < \delta$ . If

$$\xi - \delta < x < \xi < y < \xi + \delta,$$

then

$$\begin{aligned} & |[f(y) - f(x)](y - x)^{-1} - f'(\xi)| \\ &= |(y - \xi)(y - x)^{-1}[(f(y) - f(\xi))(y - \xi)^{-1} - f'(\xi)] \\ &\quad + (\xi - x)(y - x)^{-1}[(f(\xi) - f(x))(\xi - x)^{-1} - f'(\xi)]| \end{aligned}$$

$$\begin{aligned}
&\leq (y - \xi)(y - x)^{-1} |(f(y) - f(\xi))(y - \xi)^{-1} - f'(\xi)| \\
&\quad + (\xi - x)(y - x)^{-1} |(f(\xi) - f(x))(\xi - x)^{-1} - f'(\xi)| \\
&< [(y - \xi) + (\xi - x)](y - x)^{-1} \varepsilon = \varepsilon.
\end{aligned}$$

Thus, if  $f$  has a direction  $d$  at  $\xi$ , then by (1),

$$d = f'(\xi). \quad (2)$$

However,  $f$  may have no direction at  $\xi$  (take, e.g.,  $\xi = 0$  and  $f(x) = x^2$  for  $x \geq 0$ ,  $f(x) = 0$  for  $x < 0$ . If  $f$  had a direction at 0, it would have to be 0).

For the function  $f(x) \equiv |x|$ , every  $d \in (-1, 1)$  is a direction of  $f$  at 0. For set

$$a_n = (d - 1)(d + 1)^{-1} n^{-1}, \quad n = 1, 2, \dots$$

Then, for these  $n$ ,

$$[f(n^{-1}) - f(a_n)]/(n^{-1} - a_n) = d.$$

## 2

**THEOREM 1.** *Let  $f$  be a real function, continuous on some open interval containing 0 and let  $f(0) = 0$ . A necessary and sufficient condition that*

$$f'(0) = 0$$

*is*

(a)  $|f(x)| \leq K|x|$  on an open interval containing 0,  $K$  being a constant, and

(b)  $|f|$  has no direction  $\neq 0$  at 0.

*Proof. Necessity.* (a) is obvious. Since  $|f|'_x = 0$ , if  $|f|$  has a direction at 0, it is 0.

*Sufficiency.* Let  $f$  be continuous in  $[-\delta, \delta]$ ,  $\delta > 0$ . By (a),

$$-\infty < A = \lim_{x \rightarrow 0} |f(x)|/x \leq \overline{\lim}_{x \rightarrow 0} |f(x)|/x = B < \infty, \quad A \leq 0, B \geq 0.$$

Suppose the second inequality from the left is strict. Then, as is easily seen, there are sequences  $(a_n)_{n=1}^\infty$ ,  $(b_n)_{n=1}^\infty$  with

$$-\delta \leq a_n < 0, 0 < b_n \leq \delta \text{ for } n = 1, 2, \dots; a_n \rightarrow 0, b_n \rightarrow 0;$$

$$\lim_{n \rightarrow \infty} |f(a_n)|/a_n = A < B = \lim_{n \rightarrow \infty} |f(b_n)|/b_n.$$

Let  $(h_n)_{n=1}^\infty$  and  $(k_n)_{n=1}^\infty$  be subsequences of  $1, 2, \dots$ . Set, for  $n = 1, 2, \dots$ ,

$$\begin{aligned} \Delta_n &= [|f(b_{k_n})| - |f(a_{h_n})|](b_{k_n} - a_{h_n})^{-1} \\ &= b_{k_n}(b_{k_n} - a_{h_n})^{-1} |f(b_{k_n})| b_{k_n}^{-1} - a_{h_n}(b_{k_n} - a_{h_n})^{-1} |f(a_{h_n})| a_{h_n}^{-1}. \end{aligned}$$

Take, first,  $h_n \equiv n$ ,  $(k_n)_{n=1}^\infty$  so that  $b_{k_n}/a_n \rightarrow 0$  and set  $b_n^* \equiv b_{k_n}$ . Then  $\Delta_n \rightarrow A$ . Second, take  $k_n \equiv n$ ,  $(h_n)_{n=1}^\infty$  so that  $a_{h_n}/b_n \rightarrow 0$  and set  $a_n^* \equiv a_{h_n}$ . Then  $\Delta_n \rightarrow B$ . Thus

$$[|f(b_n^*)| - |f(a_n)|]/(b_n^* - a_n) \rightarrow A, \quad [|f(b_n)| - |f(a_n^*)|]/(b_n - a_n^*) \rightarrow B. \quad (3)$$

For  $n = 1, 2, \dots$ , consider the function

$$\begin{aligned} g_n(t) &\equiv [|f(b_n^* + t(b_n - b_n^*))| - |f(a_n + t(a_n^* - a_n))|]/ \\ &\quad [b_n^* + t(b_n - b_n^*) - (a_n + t(a_n^* - a_n))]. \end{aligned}$$

By (3),  $g_n(0) \rightarrow A$ ,  $g_n(1) \rightarrow B$ . Let

$$\eta \neq 0, \quad A < \eta < B.$$

For all  $n \geq$  some integer  $n_0 \geq 1$ ,

$$g_n(0) < \eta < g_n(1)$$

and as  $g_n$  is continuous in  $[0, 1]$ , there is  $t_n$ ,  $0 < t_n < 1$ , with

$$g_n(t_n) = \eta. \quad (4)$$

Set for all  $n \geq n_0$ ,

$$\alpha_n = a_n + t_n(a_n^* - a_n), \quad \beta_n = b_n^* + t_n(b_n - b_n^*). \quad (5)$$

Then for these  $n$ ,

$$\alpha_n < 0 < \beta_n, \quad [|f(\beta_n)| - |f(\alpha_n)|]/(\beta_n - \alpha_n) = \eta$$

and  $\alpha_n \rightarrow 0$ ,  $\beta_n \rightarrow 0$ . Thus  $\eta \neq 0$  is a direction of  $|f|$  at 0, contradicting (b). Therefore  $|f|$  is differentiable at 0. But then  $|f|'(0)$  must be 0. Hence  $f'(0) = 0$ .

**THEOREM 2.** Let  $\xi$  and  $C$  be real numbers and  $f$  a real function, continuous on some open interval containing  $\xi$ . A necessary and sufficient condition that

$$f'(\xi) = C$$

is

(a)  $|f(x) - f(\xi)| \leq K_1 |x - \xi|$  on an open interval containing  $\xi$ ,  $K_1$  being a constant, and

(b)  $|f(x + \xi) - f(\xi) - Cx|$  has no direction  $\neq 0$  at 0.

Thus, for a real function  $f$ , continuous on an open interval containing a point  $\xi$ ,  $f'(\xi)$  can be defined, without appeal to limit of a difference quotient, as the unique  $C$  (whenever such unique  $C$  exists) satisfying (a) and (b) of Theorem 2. (Of course, by Theorem 2, uniqueness of such a  $C$  is equivalent to its existence.)

*Proof of Theorem 2. Necessity.* (a) is obvious. Since  $|f(x + \xi) - f(\xi) - Cx|'_{x=0} = 0$ , if  $|f(x + \xi) - f(\xi) - Cx|$  has a direction at 0, it is 0.

*Sufficiency.* Set

$$\hat{f}(x) \equiv f(x + \xi) - f(\xi) - Cx. \quad (6)$$

Then the properties of  $f$  in Theorem 1, first sentence and (a), (b) of the second sentence are satisfied by  $\hat{f}$ . By Theorem 1,  $f'(\xi) = \hat{f}'(0) + C = C$ .

### 3

For strictly convex functions, Theorem 2 can be simplified:

**THEOREM 3.** *Let  $\xi$  and  $C$  be real numbers and  $f$  a real function, strictly convex in an open interval  $I$  containing  $\xi$ . A necessary and sufficient condition that*

$$f'(\xi) = C \quad (7)$$

*is that  $C$  is the unique direction of  $f$  at  $\xi$ . In fact, if  $f$  has no direction  $\neq C$  at  $\xi$ , then  $f'(\xi) = C$  and  $C$  is a direction of  $f$  at  $\xi$ .*

The second sentence of Theorem 3 provides a simple definition of derivative of a strictly convex function.

*Proof of Theorem 3.* Suppose (7). Let  $J \ni \xi$  be an open interval. Choose in  $I \cap J$  any points  $a < \xi < b$ . Then

$$[f(\xi) - f(a)]/(\xi - a) < f'(\xi) < [f(b) - f(\xi)]/(b - \xi).$$

By continuity, if  $a^*$  and  $b^*$  are in  $I \cap J$ , sufficiently close to  $\xi$  and  $a^* < \xi < b^*$ , then

$$[f(b^*) - f(a)]/(b^* - a) < f'(\xi) < [f(b) - f(a^*)]/(b - a^*). \quad (8)$$

Set

$$g(t) \equiv [f(b^* + t(b - b^*)) - f(a + t(a^* - a))] / [b^* + t(b - b^*) - (a + t(a^* - a))]. \quad (9)$$

It is continuous in  $[0, 1]$  and  $g(0) < f'(\xi) < g(1)$ . Hence there is a  $t \in (0, 1)$  with  $g(t) = f'(\xi)$ . This, together with the sentence of (2) proves that  $C$  is the unique direction of  $f$  at  $\xi$ .

Suppose now  $f$  has no direction  $\neq C$  at  $\xi$ . We show  $f'(\xi)$  exists. Suppose not. Let<sup>1</sup>

$$f'_-(\xi) < \eta < f'_+(\xi), \quad \eta \neq C.$$

Let  $J \ni \xi$  be an open interval. Then clearly there are in  $I \cap J$  points  $a < \xi < b^*$ ,  $a^* < \xi < b$  satisfying (8) with  $\eta$  replacing  $f'(\xi)$ . Consider (9). As  $g(0) < \eta < g(1)$ , there is  $t \in (0, 1)$  for which  $g(t) = \eta$ , so that  $\eta$  is a direction of  $f$  at  $\xi$ , a contradiction. By the necessity of (7),  $f'(\xi)$  is a direction of  $f$  at  $\xi$ . Hence  $f'(\xi) = C$ .

#### 4

**THEOREM 4.** Let  $\xi$  be a real number and  $f$  a real function. A necessary and, if  $f$  is continuous on some open interval  $I$  containing  $\xi$ , also a sufficient condition for the differentiability of  $f$  at  $\xi$  is (a) of Theorem 2 and that  $f$  has at most one direction at  $\xi$ .

*Proof.* Necessity is obvious. To prove sufficiency suppose, on the contrary,

$$-\infty < \varliminf_{x \rightarrow \xi} [f(x) - f(\xi)]/(x - \xi) < \overline{\lim}_{x \rightarrow \xi} [f(x) - f(\xi)]/(x - \xi) < \infty.$$

Then there are sequences  $(a_n)_{n=1}^\infty$ ,  $(b_n)_{n=1}^\infty$  and numbers  $\alpha, \beta$  with  $a_n \in I$ ,  $a_n < \xi$ ,  $b_n \in I$ ,  $b_n > \xi$  for  $n = 1, 2, \dots$ ;  $a_n \rightarrow \xi$ ,  $b_n \rightarrow \xi$ :

$$\lim_{n \rightarrow \infty} [f(a_n) - f(\xi)]/(a_n - \xi) = \alpha \neq \beta = \lim_{n \rightarrow \infty} [f(b_n) - f(\xi)]/(b_n - \xi).$$

Let  $(h_n)_{n=1}^\infty$  and  $(k_n)_{n=1}^\infty$  be subsequences of  $1, 2, \dots$ . Set, for  $n = 1, 2, \dots$ ,

$$\begin{aligned} \Delta_n &= [f(b_{k_n}) - f(a_{h_n})](b_{k_n} - a_{h_n})^{-1} \\ &= (b_{k_n} - \xi)[(b_{k_n} - \xi) - (a_{h_n} - \xi)]^{-1} [f(b_{k_n}) - f(\xi)](b_{k_n} - \xi)^{-1} \\ &\quad - (a_{h_n} - \xi)[(b_{k_n} - \xi) - (a_{h_n} - \xi)]^{-1} [f(a_{h_n}) - f(\xi)](a_{h_n} - \xi)^{-1}. \end{aligned}$$

Take, first,  $h_n \equiv n$ ,  $(k_n)_{n=1}^\infty$  so that  $(b_{k_n} - \xi)/(a_n - \xi) \rightarrow 0$  and set  $h_n^* \equiv k_n$ .

<sup>1</sup>  $f'_-(\xi)$  and  $f'_+(\xi)$  are, respectively, the left and right derivative of  $f$  at  $\xi$ .

Then  $\Delta_n \rightarrow \alpha$ . Second, take  $k_n \equiv n$ ,  $(h_n)_{n=1}^\infty$  so that  $(a_{h_n} - \xi)/(b_n - \xi) \rightarrow 0$  and set  $a_n^* \equiv a_{h_n}$ . Then  $\Delta_n \rightarrow \beta$ . Thus

$$[f(b_n^*) - f(a_n)]/(b_n^* - a_n) \rightarrow \alpha, \quad [f(b_n) - f(a_n^*)]/(b_n - a_n^*) \rightarrow \beta. \quad (10)$$

Let

$$\min(\alpha, \beta) < \eta < \max(\alpha, \beta). \quad (11)$$

For  $n = 1, 2, \dots$ , consider the function

$$g_n(t) \equiv [f(b_n^* + t(b_n - b_n^*)) - f(a_n + t(a_n^* - a_n))]/[b_n^* + t(b_n - b_n^*) - (a_n + t(a_n^* - a_n))].$$

By (10),

$$g_n(0) \rightarrow \alpha, \quad g_n(1) \rightarrow \beta.$$

For all  $n \geq$  some integer  $n_0 \geq 1$ ,  $\eta$  is strictly between  $g_n(0)$  and  $g_n(1)$  and as  $g_n$  is continuous in  $[0, 1]$ , there is  $t_n \in (0, 1)$  with (4).

Set, for all  $n \geq n_0$ , (5). Then for these  $n$ ,

$$\alpha_n < \xi < \beta_n, \quad [f(\beta_n) - f(\alpha_n)]/(\beta_n - \alpha_n) = \eta$$

and  $\alpha_n \rightarrow \xi$ ,  $\beta_n \rightarrow \xi$ . Thus *every*  $\eta$  satisfying (11) is a direction of  $f$  at  $\xi$ , contradicting hypothesis.

## 5

We conclude with some remarks:

I. The theorem in the second paragraph of Section I has the following converse: Let  $\xi$  be a real number and  $f$  a real function for which  $[f(y) - f(x)]/(y - x)$  converges as  $x \rightarrow \xi^-$ ,  $y \rightarrow \xi^+$ , say, to  $C$ . Then  $C = f'(\xi)$  provided we take  $f(\xi)$  to be  $\lim_{x \rightarrow \xi} f(x)$  which clearly exists and is finite. For let  $\varepsilon > 0$  and let  $\delta > 0$  be such that

$$|[f(y) - f(x)](y - x)^{-1} - C| < \varepsilon \quad (12)$$

whenever  $\xi - \delta < x < \xi < y < \xi + \delta$ . If  $\xi < y < \xi + \delta$ , then, letting  $x \rightarrow \xi$  in (12),

$$|[f(y) - f(\xi)](y - \xi)^{-1} - C| \leq \varepsilon.$$

Similarly, if  $\xi - \delta < x < \xi$ , then

$$|[f(\xi) - f(x)](\xi - x)^{-1} - C| \leq \varepsilon.$$

Thus, if  $0 < |x - \xi| < \delta$ , then

$$|[f(x) - f(\xi)](x - \xi)^{-1} - C| \leq \varepsilon.$$

II. Similarly, it is an elementary fact that if  $\xi$  is a real number and  $f$  a real function, then

$$\lim_{x \rightarrow \xi} [f(x) - f(\xi)]/(x - \xi) = \infty \quad (13)$$

implies

$$\lim_{\substack{x \rightarrow \xi^- \\ y \rightarrow \xi^+}} [f(y) - f(x)]/(y - x) = \infty$$

and, provided  $f$  is continuous at  $\xi$ , also conversely; in particular, (13) implies that  $f$  has no direction at  $\xi$ . Clearly,  $\infty$  can be replaced by  $-\infty$ .

III. In Theorem 1(b),  $|f|$  cannot be replaced by  $f$ . For let  $f(x) = x$  whenever  $x < 0$ ,  $f(x) = x + x^2$  whenever  $x \geq 0$ . Then  $f'(0) = 1$ , and so, by the sentence of (2), if  $f$  had a direction at 0, it would be 1. But 1 is not a direction of  $f$  at 0 because if  $a < 0 < b$ , then  $[f(b) - f(a)]/(b - a) > 1$ . Thus  $|f(x)| \leq 2|x|$  on  $(-1, 1)$  and  $f$  has no direction at 0, but  $f'(0) \neq 0$ .

IV. Theorem 4 remains true if its continuity hypothesis is replaced by: (\*) *There is an open interval  $I$  containing  $\xi$  on which  $f$  is defined such that if*

$$a' < \xi < b', \quad a'' < \xi < b''; \quad a', b', a'', b'' \in I,$$

*and  $\eta$  is strictly between  $[f(b') - f(a')]/(b' - a')$  and  $[f(b'') - f(a'')]/(b'' - a'')$ , then there are  $a, b$  with*

$$\min(a', a'') \leq a < \xi < b \leq \max(b', b''), \\ [f(b) - f(a)]/(b - a) = \eta.$$

Indeed, to prove sufficiency, follow the proof of Theorem 4 through (11). For all  $n \geq$  some integer  $n_0 \geq 1$ , we have

$$a_n < \xi < b_n^*, \quad a_n^* < \xi < b_n; \quad a_n, b_n^*, a_n^*, b_n \in I$$

and  $\eta$  is strictly between  $[f(b_n^*) - f(a_n)]/(b_n^* - a_n)$  and  $[f(b_n) - f(a_n^*)]/(b_n - a_n^*)$ , and therefore, by (\*), there are  $\alpha_n, \beta_n$  with

$$\min(a_n, a_n^*) \leq \alpha_n < \xi < \beta_n \leq \max(b_n^*, b_n) \text{ and hence } \alpha_n \rightarrow \xi, \beta_n \rightarrow \xi;$$

$$[f(\beta_n) - f(\alpha_n)]/(\beta_n - \alpha_n) = \eta.$$

Thus *every*  $\eta$  satisfying (11) is a direction of  $f$  at  $\xi$ , contradicting hypothesis.

Clearly Theorem 4, both in its original and modified version, remains true if “ $f$  has at most one direction at  $\xi$ ” is replaced by “the set of directions of  $f$  at  $\xi$  does not include any open interval.”

Of course, as one sees from the proof of (the original) Theorem 4, the continuity hypothesis of that theorem implies (\*) with the same  $I$ .

V. In Theorem 2, the continuity hypothesis is not used in proving necessity. Also, analogously to IV, one readily sees that this continuity hypothesis can be replaced by the following condition, analogous to (\*) of IV but concerning  $\hat{f}$  of (6): (\*\*) *There is an open interval  $J$  containing 0 on which  $\hat{f}$  is defined such that if*

$$a' < 0 < b', \quad a'' < 0 < b''; \quad a', b', a'', b'' \in J$$

*and*

$$[|\hat{f}(b')| - |\hat{f}(a')|]/(b' - a') < \eta < [|\hat{f}(b'')| - |\hat{f}(a'')|]/(b'' - a''),$$

*then there are  $a, b$  with*

$$\min(a', a'') \leq a < 0 < b \leq \max(b', b''),$$

$$[|\hat{f}(b)| - |\hat{f}(a)|]/(b - a) = \eta.$$

Again, the continuity hypothesis of Theorem 2 implies (\*\*). Furthermore, Theorem 2 with the continuity hypothesis replaced by (\*\*) can be used to define derivative for a very general class of functions without any appeal to limit, either in the definition of the derivative itself or in the definition of that class.

One readily sees that Theorem 2, both in its original and modified version, remains true if (b) of that theorem is replaced by “the set of directions of  $|\hat{f}|$  at 0 does not include any open interval.”

VI. Clearly, in Theorem 3, “strictly convex” can be replaced by “strictly concave.”